Introduction to the

Spectral-infinite-element method

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Outline

- Background
- Infinite-element method
- Spectral-element method
- Examples

Global problems

- Wave propagation
- Postearthquake relaxation
- Glacial isostatic adjustment
- Tidal loading

Governing equations

- Conservation of linear momentum

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\boldsymbol{\nabla}(\rho\,\mathbf{u}\cdot\mathbf{g})-\boldsymbol{\nabla}\cdot(\rho\,\mathbf{u})\,\mathbf{g}-\rho\,\boldsymbol{\nabla}\phi=\rho\,\ddot{\mathbf{u}}+2\,\rho\,\boldsymbol{\Omega}\times\dot{\mathbf{u}}+\mathbf{f}$$

- Conservation of angular momentum

$$\dot{\mathbf{H}}+\boldsymbol{\Omega}\times\mathbf{H}=\mathbf{0}$$

- Gravity potential

$$\nabla^2\phi+4\,\pi\,G\,\boldsymbol{\nabla}\cdot(\rho\,\mathbf{s})\;=\;0$$

SPECFEM3D Cartesian

- Conservation of linear momentum

$$\nabla \cdot \boldsymbol{\sigma} + \nabla(\rho\,\mathbf{u}\cdot\mathbf{g}) - \nabla\cdot(\rho\,\mathbf{u})\,\mathbf{g} - \rho\,\nabla\phi = \rho\,\ddot{\mathbf{u}} + 2\,\rho\,\boldsymbol{\Omega}\times\dot{\mathbf{u}} + \mathbf{f}$$

- Conservation of angular momentum

$$\dot{\mathbf{H}} + \boldsymbol{\Omega}\times\mathbf{H} = \mathbf{0}$$

- Gravity potential

$$\nabla^2\phi + 4\,\pi\,G\,\nabla\cdot(\rho\,\mathbf{s}) = 0$$

SPECFEM3D Globe

- Conservation of linear momentum

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{\nabla}(\rho\, \mathbf{u} \cdot \mathbf{g}) - \boldsymbol{\nabla} \cdot (\rho\, \mathbf{u})\, \mathbf{g} - \rho\, \boldsymbol{\nabla} \phi = \rho\, \ddot{\mathbf{u}} + 2\, \rho\, \boldsymbol{\Omega} \times \dot{\mathbf{u}} + \mathbf{f}$$

- Conservation of angular momentum

$$\dot{\mathbf{H}} + \boldsymbol{\Omega} \times \mathbf{H} = \mathbf{0}$$

- Gravity potential

$$\nabla^2 \phi + 4\pi G\, \boldsymbol{\nabla} \cdot (\rho\, \mathbf{s}) = 0$$

Motivation

- Conservation of linear momentum

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{\nabla}(\rho\, \mathbf{u} \cdot \mathbf{g}) - \boldsymbol{\nabla} \cdot (\rho\, \mathbf{u})\, \mathbf{g} - \rho\, \boldsymbol{\nabla} \phi = \rho\, \ddot{\mathbf{u}} + 2\, \rho\, \boldsymbol{\Omega} \times \dot{\mathbf{u}} + \mathbf{f}$$

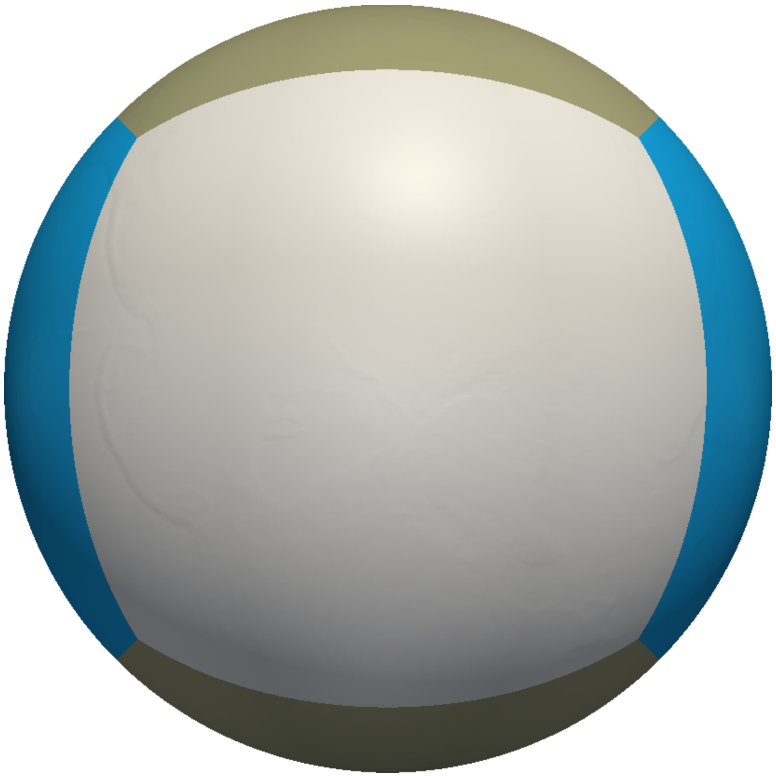
- Conservation of angular momentum

$$\dot{\mathbf{H}} + \boldsymbol{\Omega} \times \mathbf{H} = \mathbf{0}$$

- Gravity potential

$$\nabla^2 \phi + 4 \pi G\, \boldsymbol{\nabla} \cdot (\rho\, \mathbf{s}) = 0$$

Problem size



- NEX = 256
- N_{spec} = 4,450,304
- N_{node} = 299,068,155

Problem size

Explicit scheme

$$\ddot{\mathbf{s}} = \mathbf{M}^{-1}(\mathbf{f} - \mathbf{C}\,\dot{\mathbf{s}} - \mathbf{K}\,\mathbf{s})$$

- N_{dof} = 1,196,272,620
- Maximum array size = 1,196,272,620

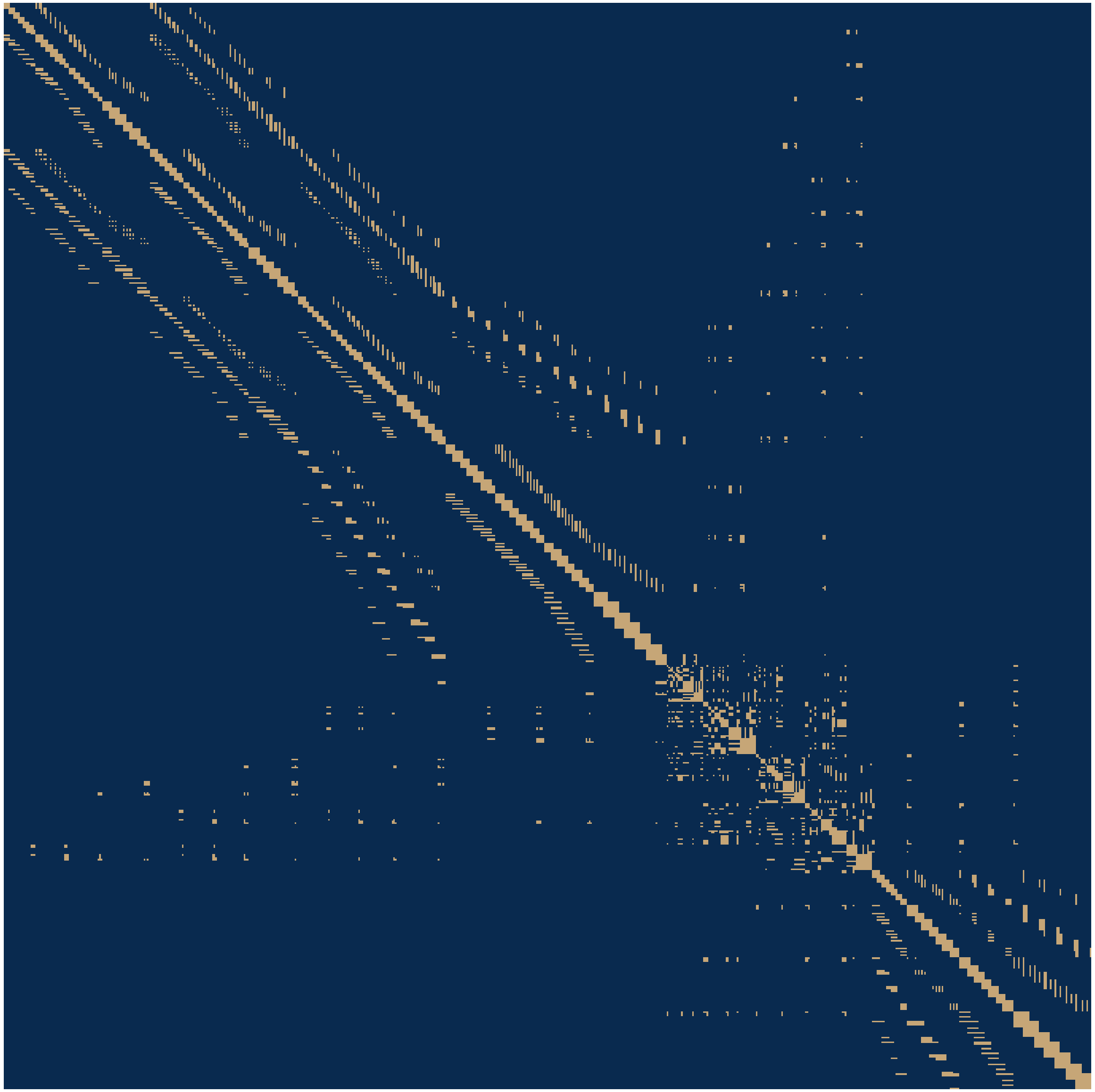
Implicit scheme

$$\mathbf{s} = \mathbf{K}^{-1}\,\mathbf{f}$$

- N_{dof} = 1,196,272,620
- Maximum array size = 1,196,272,620 X 1,196,272,620

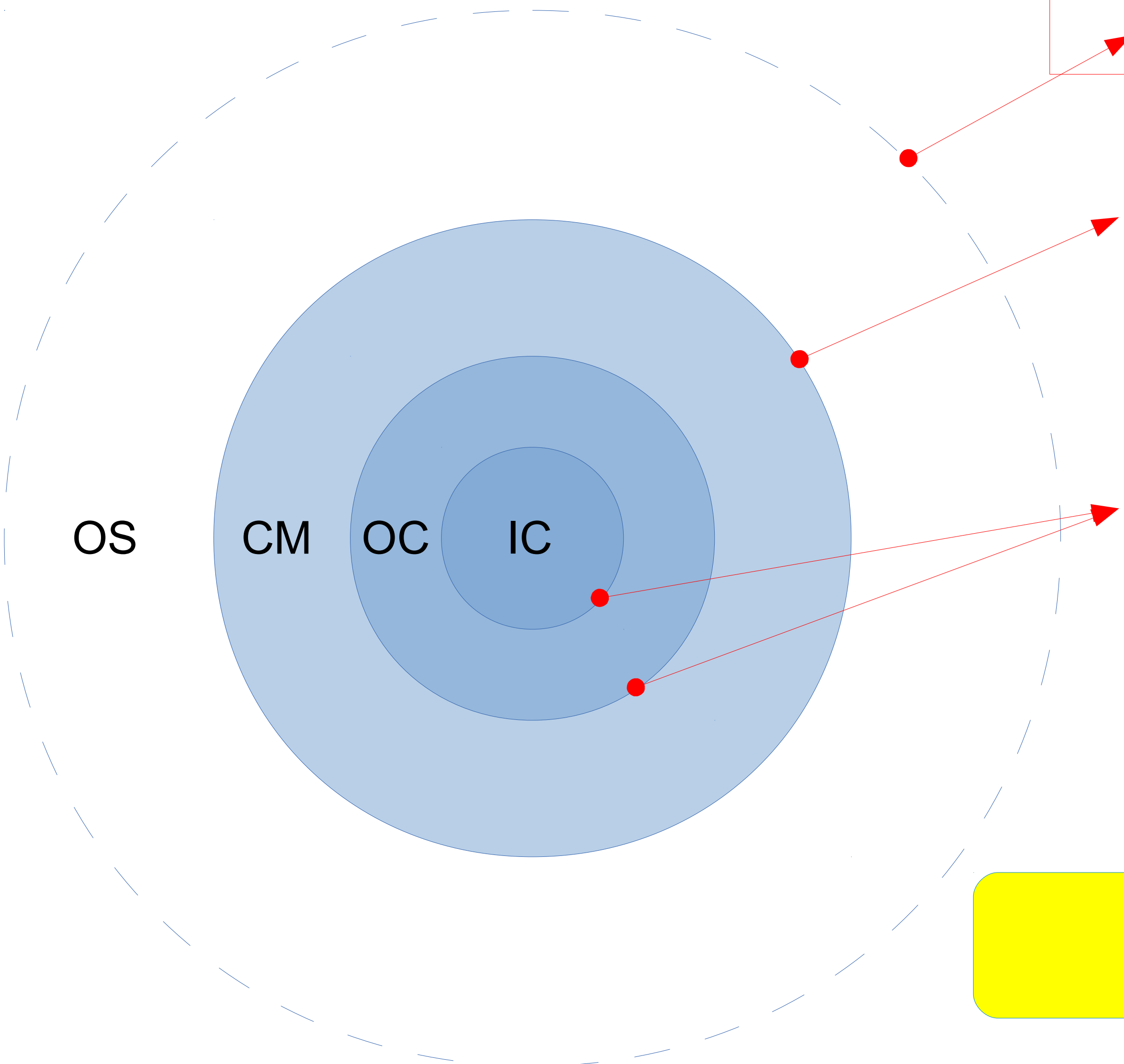
Huge problem!

Problem size



- Sparsity
- Element-by-element

Gravity



$$\phi = 0 \quad \text{at} \quad \infty$$

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \mathbf{t}$$

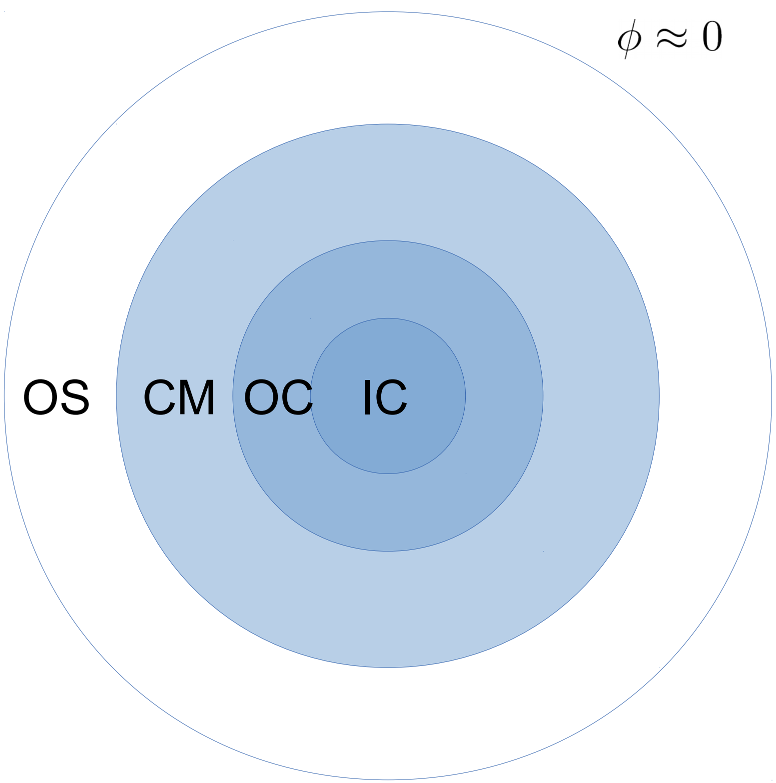
$$\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = -p^{\mathrm{f}}\, \hat{\mathbf{n}}$$

$$\mathbf{u} \cdot \hat{\mathbf{n}} = \mathbf{u}^{\mathrm{f}} \cdot \hat{\mathbf{n}}$$

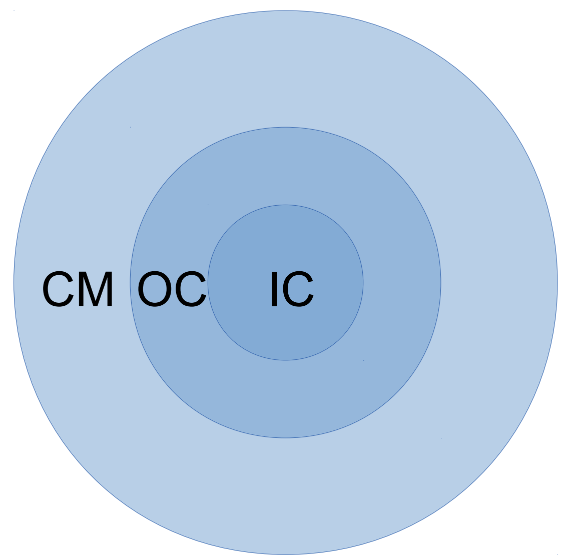
Unbounded domain

Existing methods

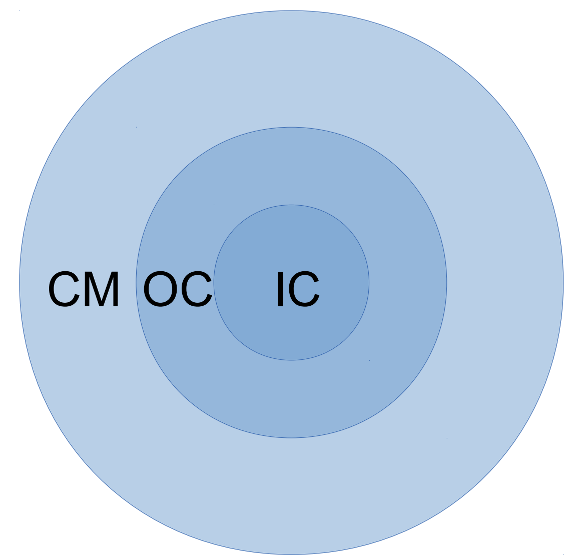
Domain extension



Spherical harmonics



First principle

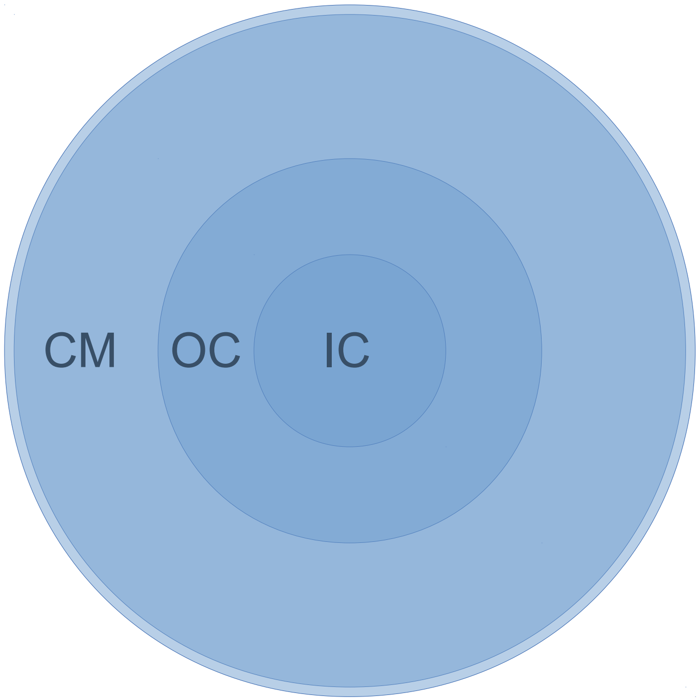


$$\phi_s = -G\int \frac{\rho(\mathbf{r})[\mathbf{s}(\mathbf{r})\cdot(\mathbf{r}_s-\mathbf{r})]}{|\mathbf{r}_s-\mathbf{r}|^3}\mathrm{d}\Omega - G\int_{r_s=R}\frac{\sigma(\mathbf{r}_s)}{|\mathbf{r}_s-\mathbf{r}|}\mathrm{d}S - G\int_{r_s=R_{\mathrm{OC}}}\frac{\sigma(\mathbf{r}_s)}{|\mathbf{r}_s-\mathbf{r}|}\mathrm{d}S - G\int_{r_s=R_{\mathrm{IC}}}\frac{\sigma(\mathbf{r}_s)}{|\mathbf{r}_s-\mathbf{r}|}\mathrm{d}S$$

Is it possible to consider full 3D model?

Infinite-element method

(Zienkiewicz et al, 1983)



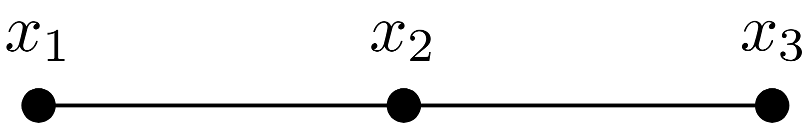
Infinite-element layer

Widely used in solid/fluid mechanics!

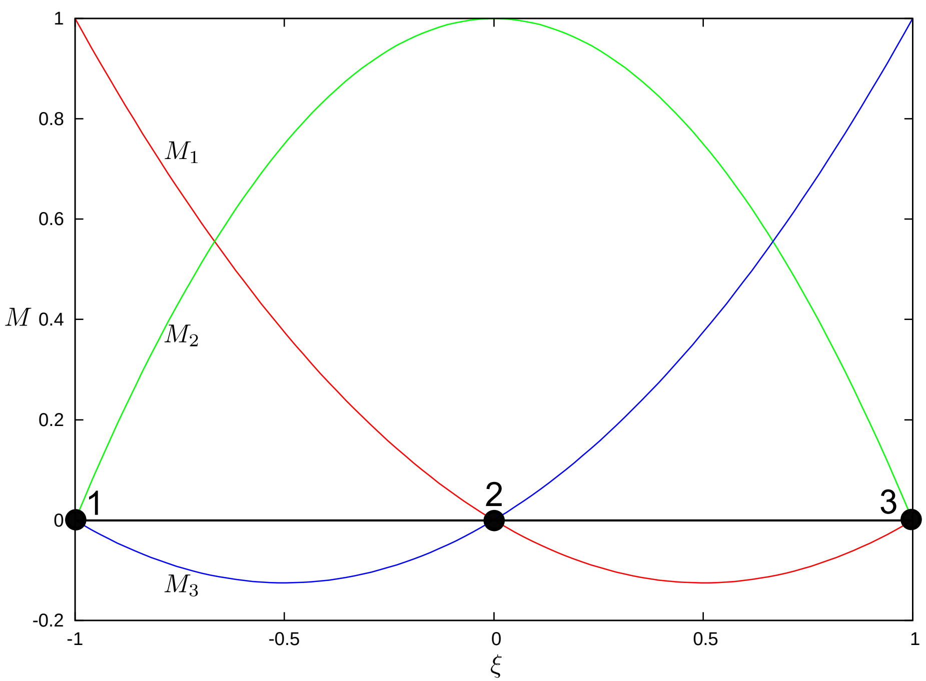
Infinite-element mapping

(Zienkiewicz et al, 1983)

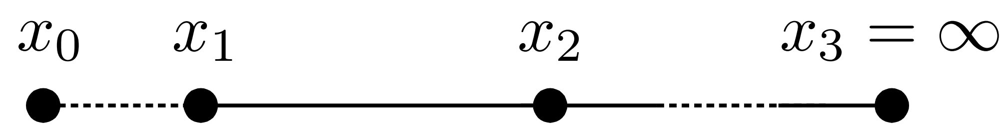
Finite element



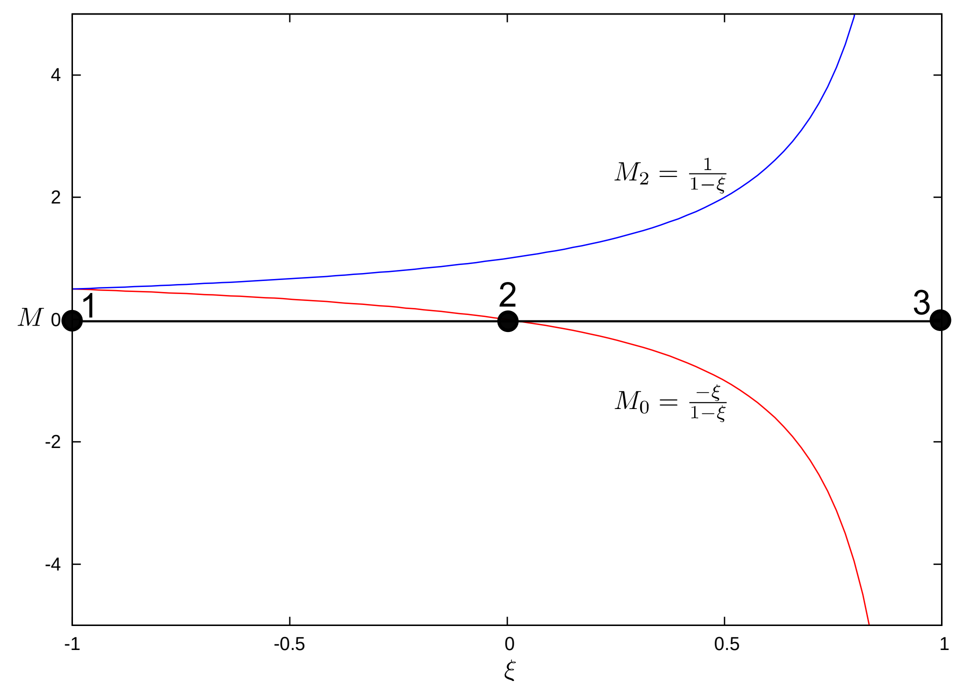
$$x = \sum_{i=1}^{3} M_i(\xi)\, x_i$$



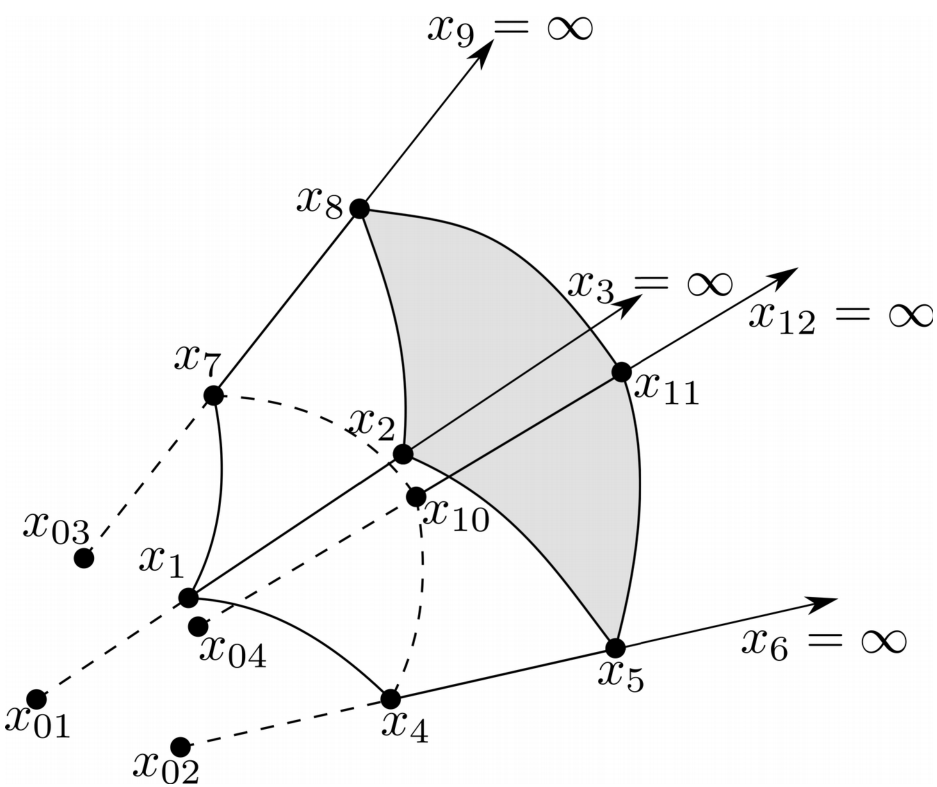
Infinite element

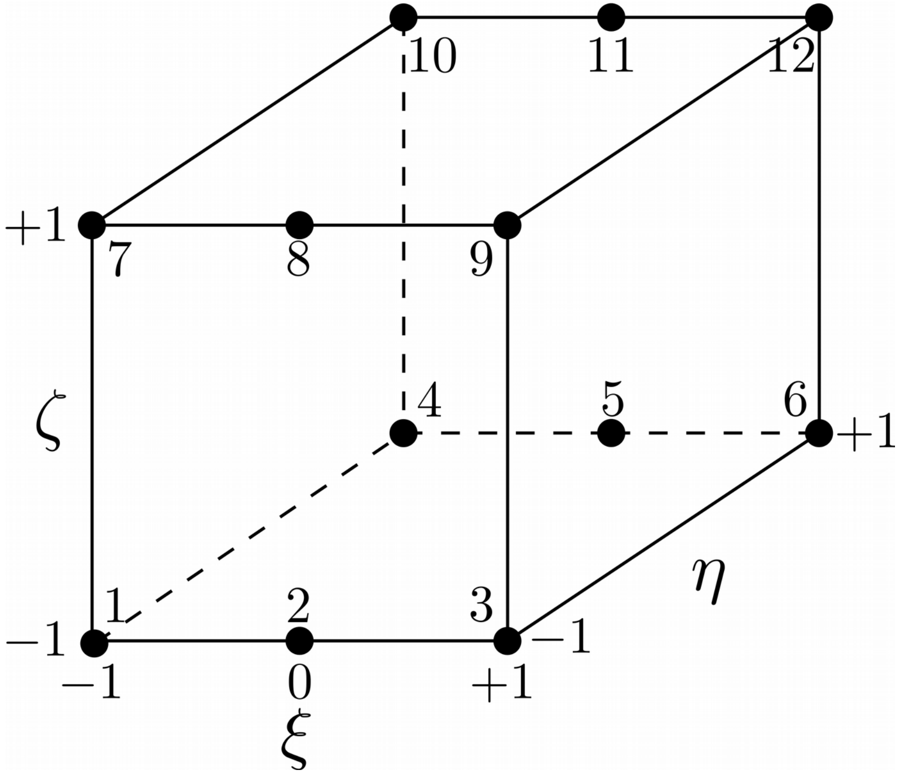


$$x = M_0(\xi)\, x_0 + M_2(\xi)\, x_2$$



3D infinite element





Why it works?

$$\phi(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \cdots$$

$$x = M_0(\xi)\,x_0 + M_2(\xi)\,x_2$$
$$r = x - x_0$$

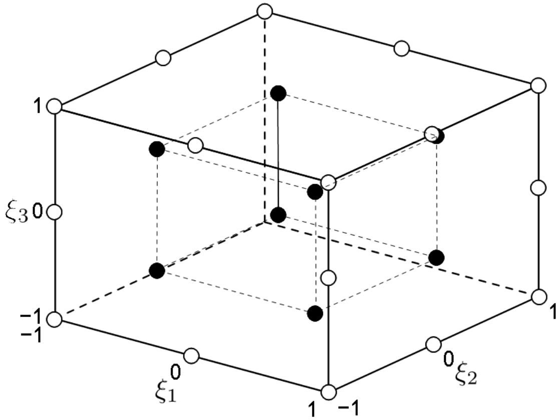
$$\phi(r) = b_0 + \frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \cdots$$

ϕ decays to b_0 as r tends to infinity

Spectral-element method

(e.g., Patera 1984, Cohen et al 1993, Faccioli et al 1997, Komatitsch & Tromp 1999)

FEM

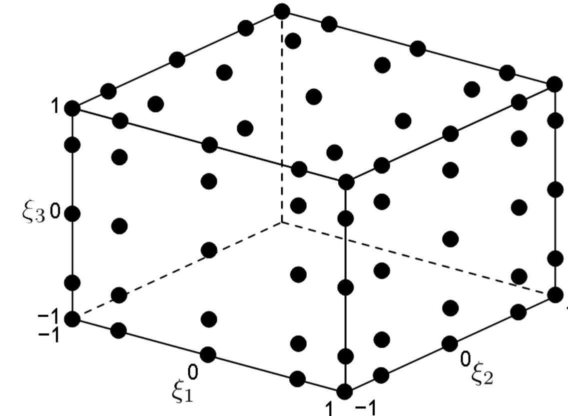


$N_i(_j) \neq \delta_{ij}$

○ Interpolation

● Integration

SEM



$N_i(\boldsymbol{\xi}_j) = \delta_{ij}$ orthogonal

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

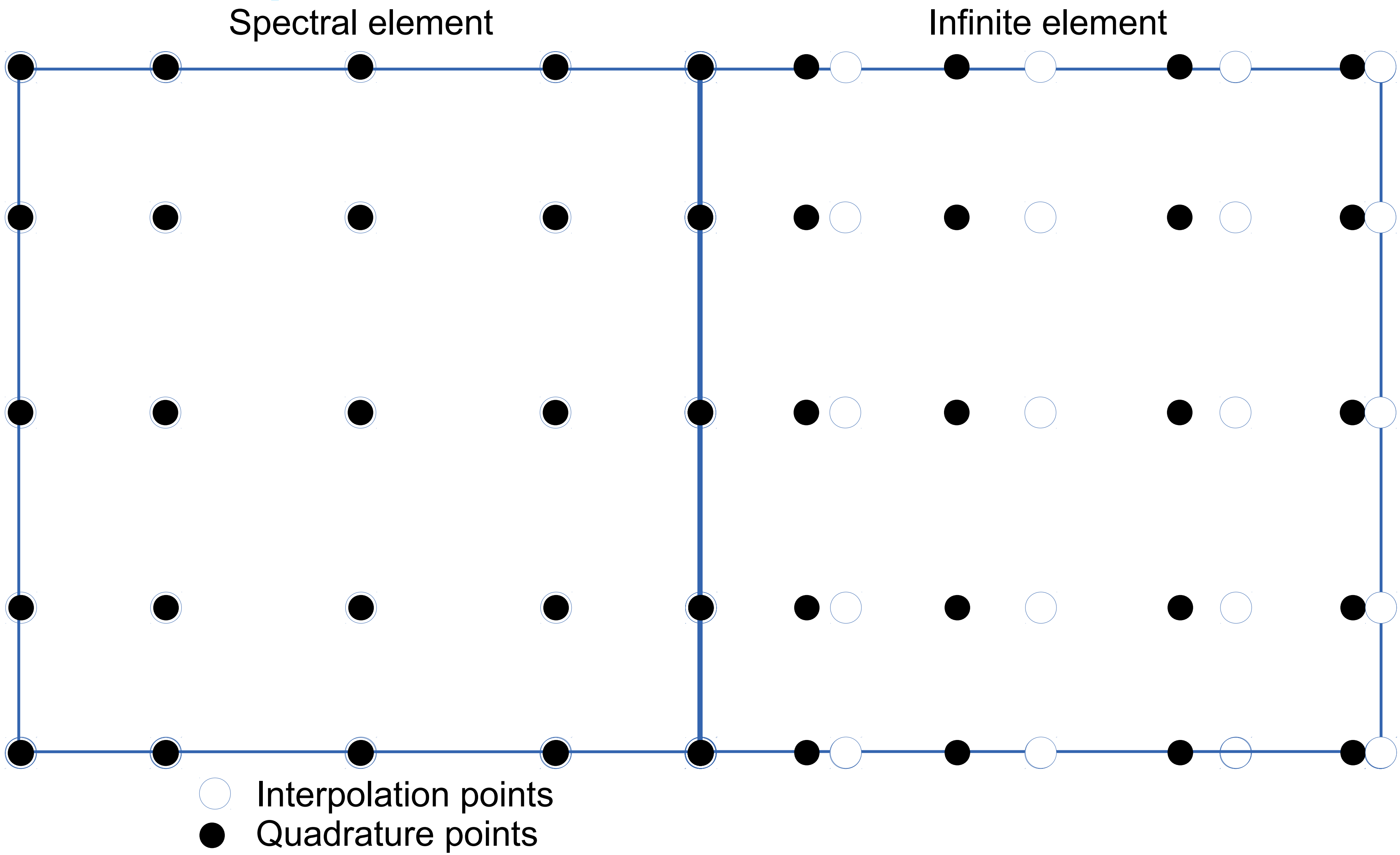
Mass matrix

$$\mathbf{M} = \int_\Omega \rho \, N_i \, N_j \, \mathrm{d}\Omega$$

$$\begin{bmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{bmatrix}$$

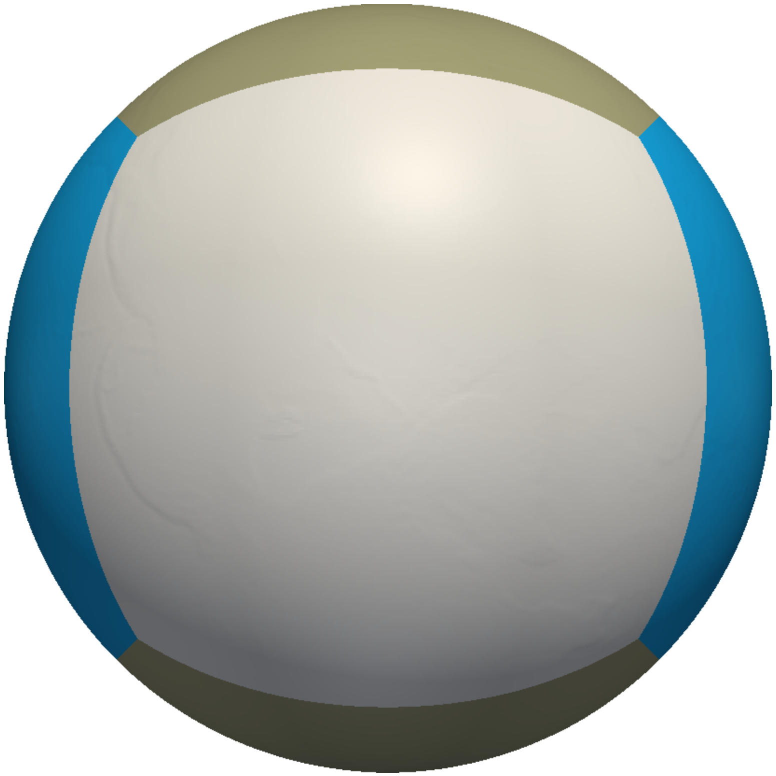
Flexibility of FEM and stability of spectral method!

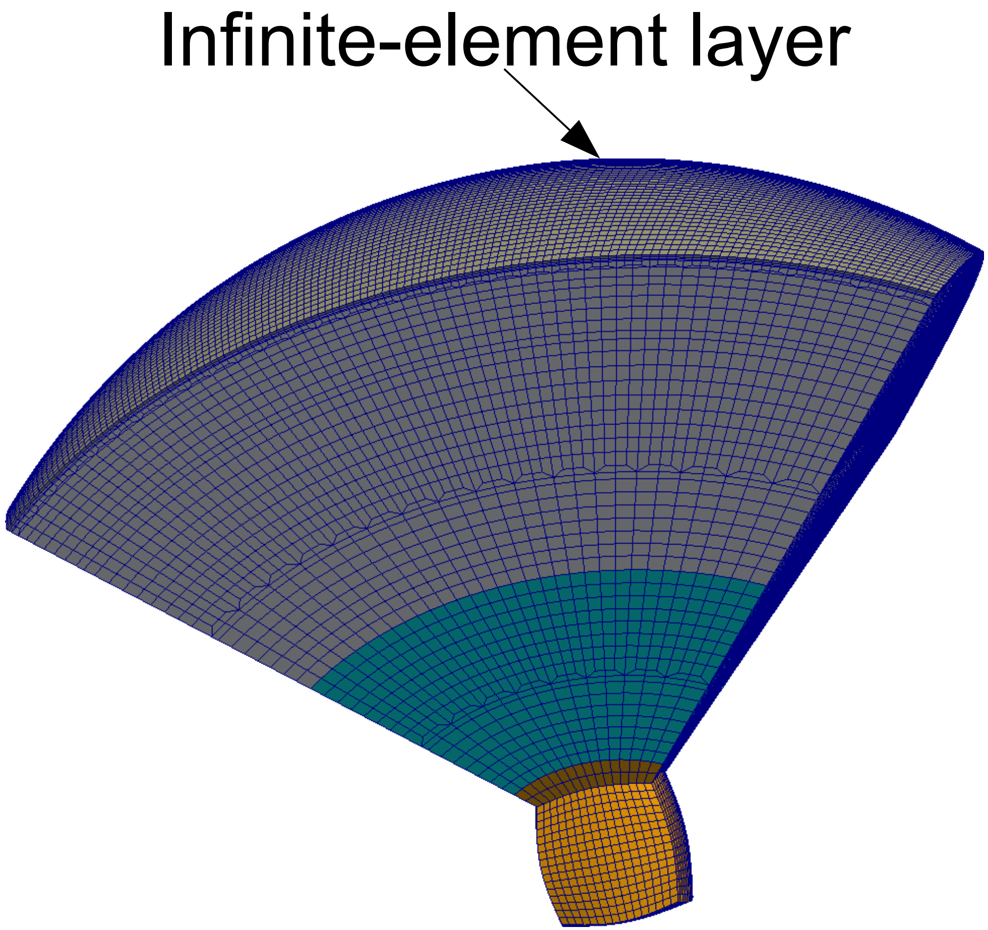
Spectral-infinite-elements



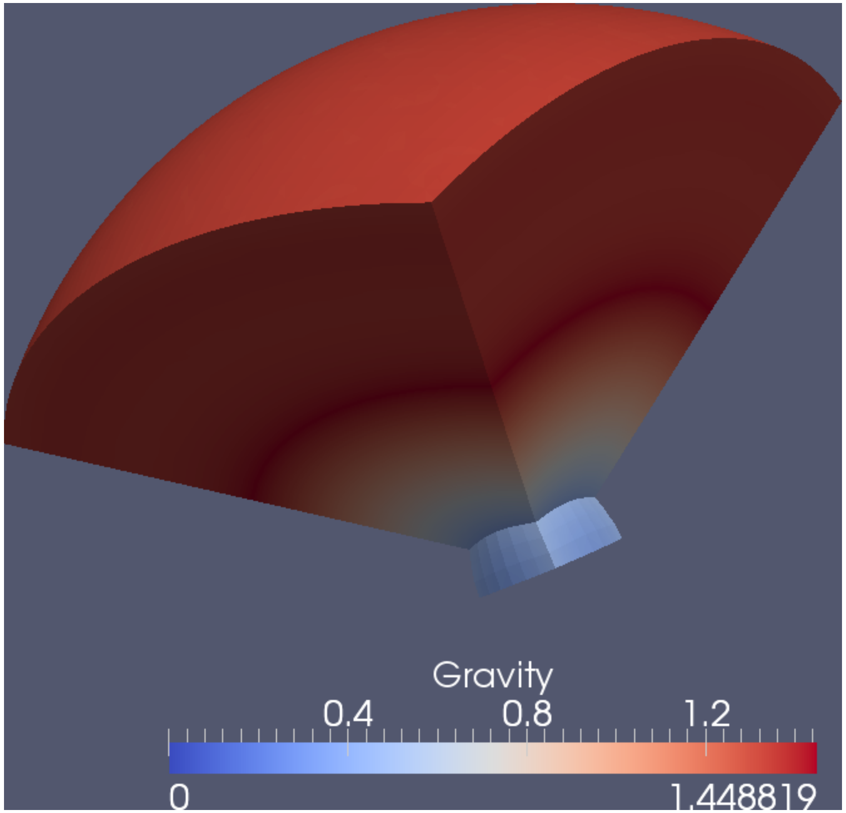
Lobatto quadrature vs Radau quadrature

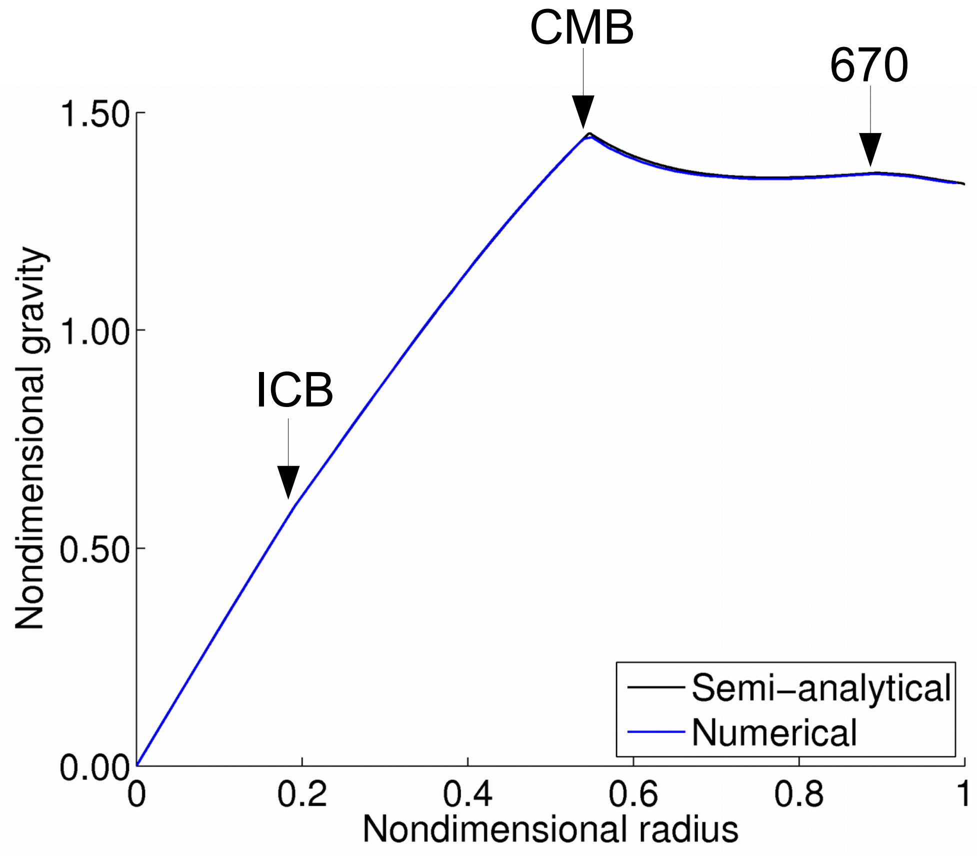
Meshing



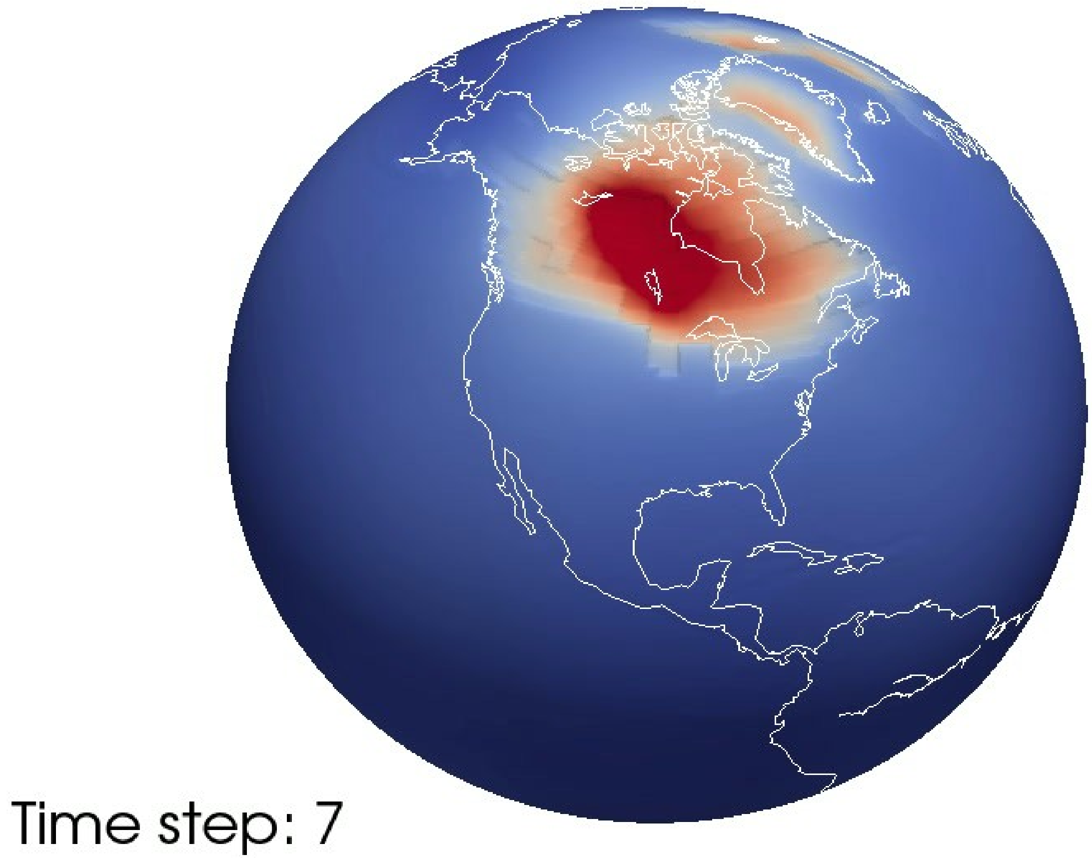


Background gravity



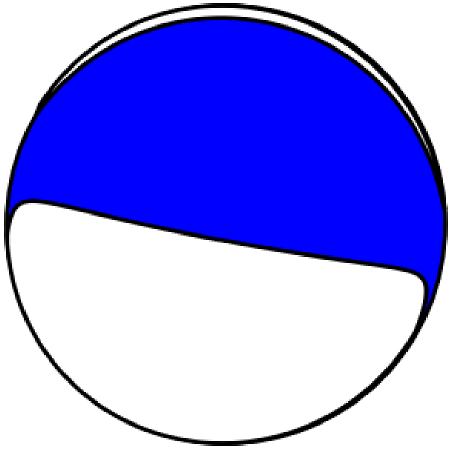


Glacial rebound

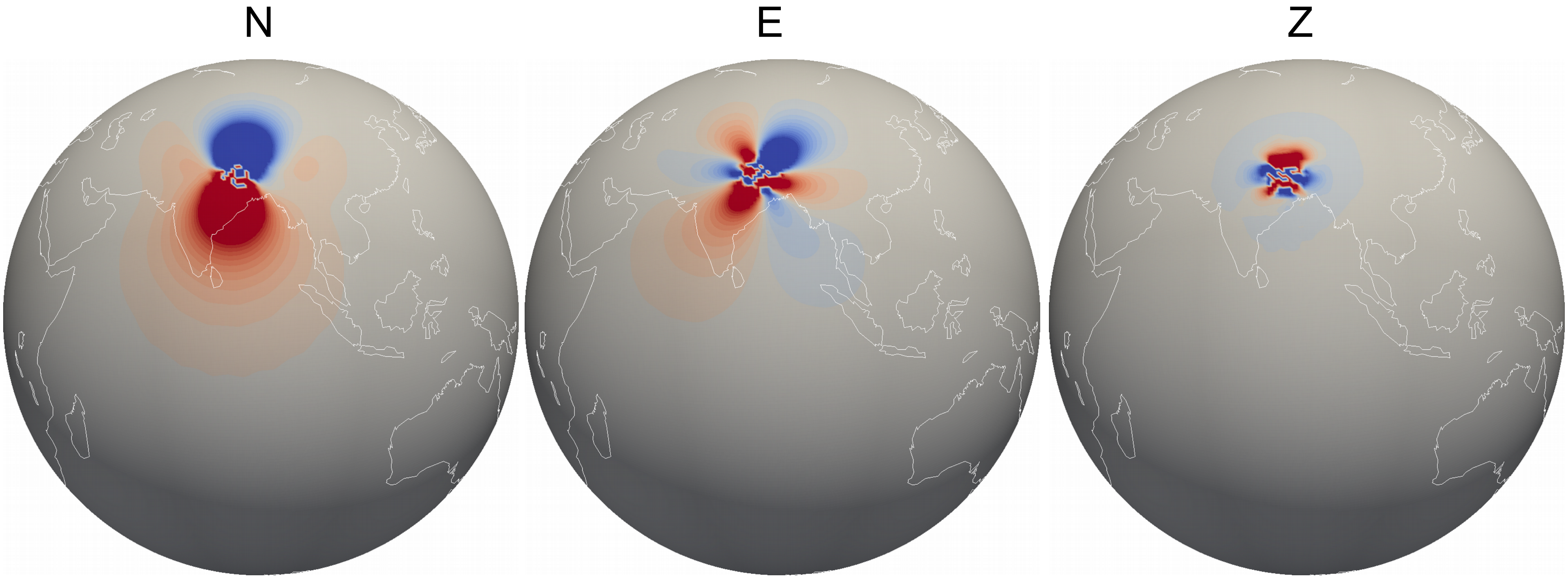


Tidal loading

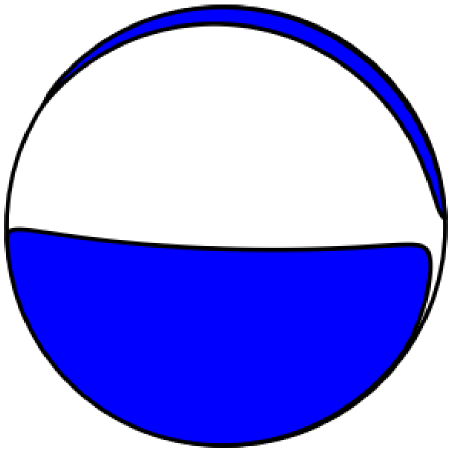
Postearthquake relaxation



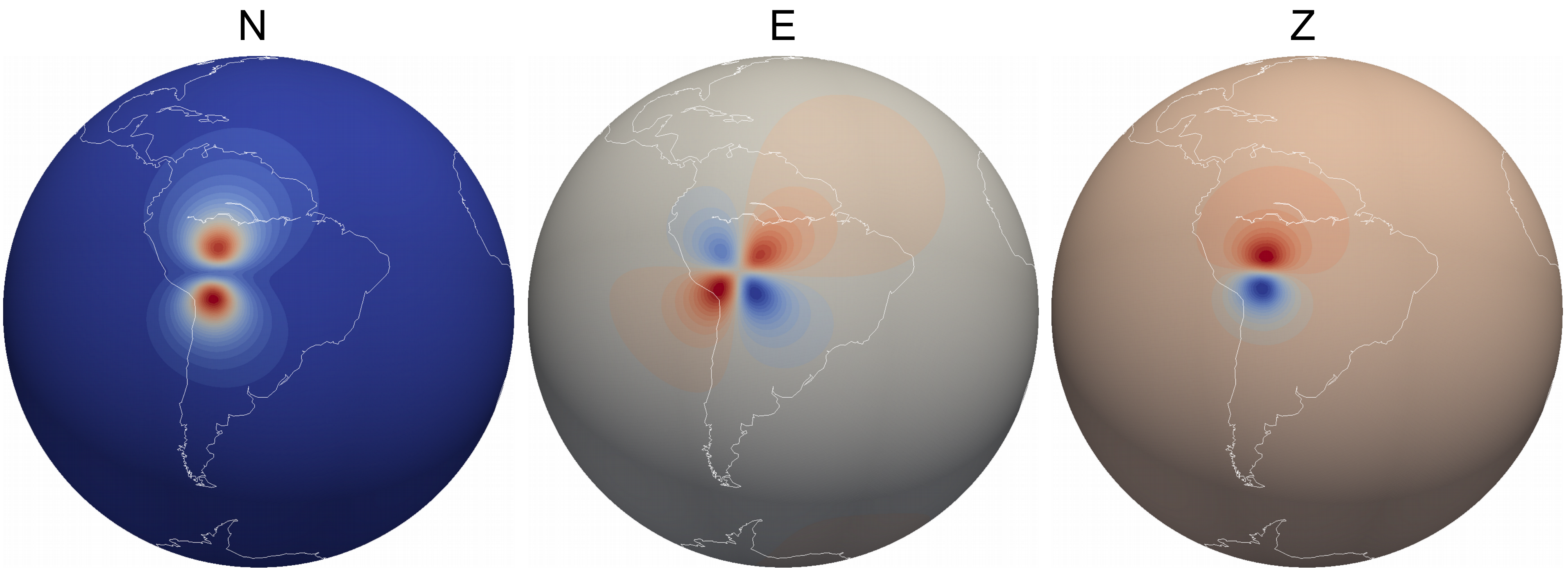
- Nepal Earthquake
- April 25, 2015
- 7.8 Mw



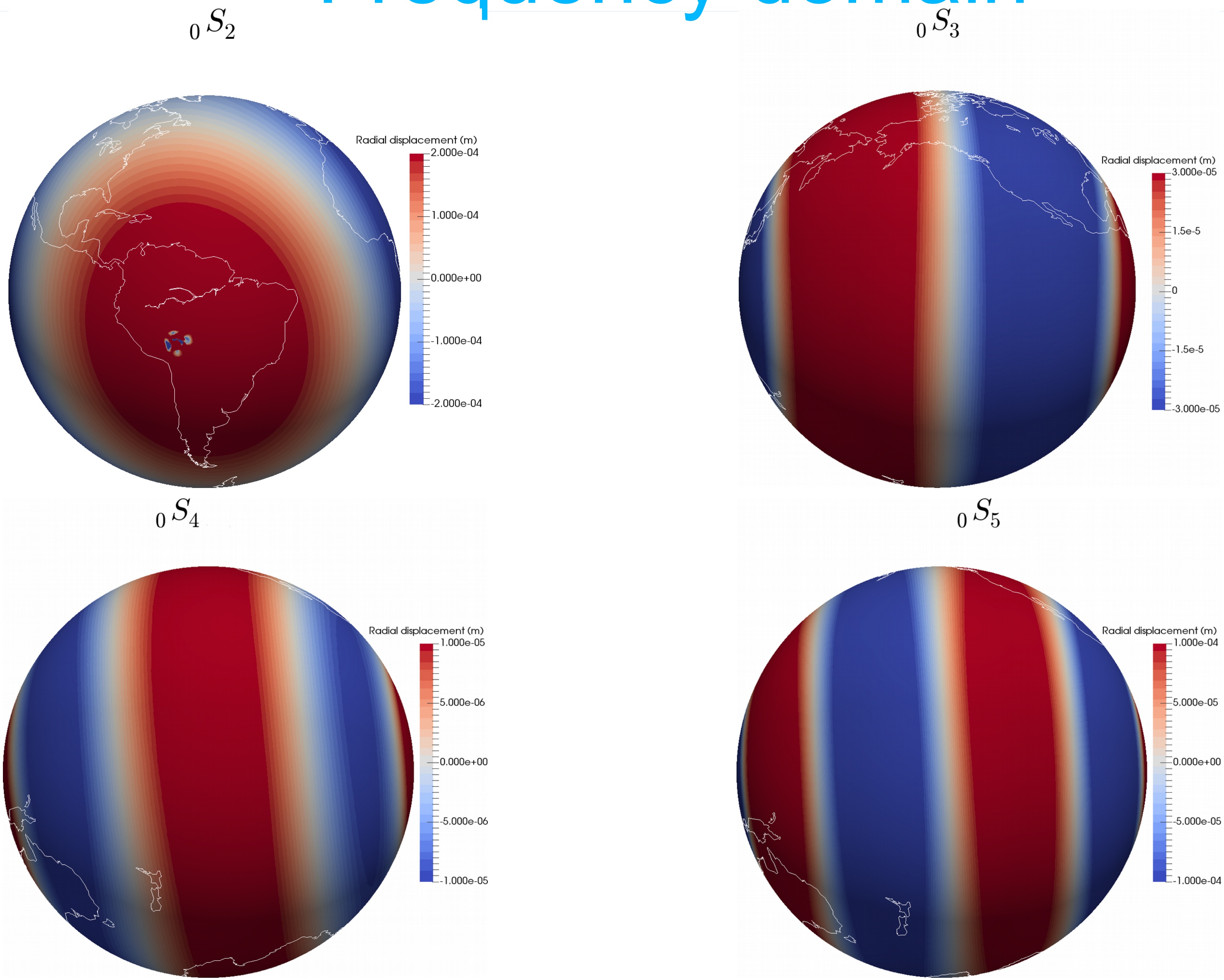
Postearthquake relaxation



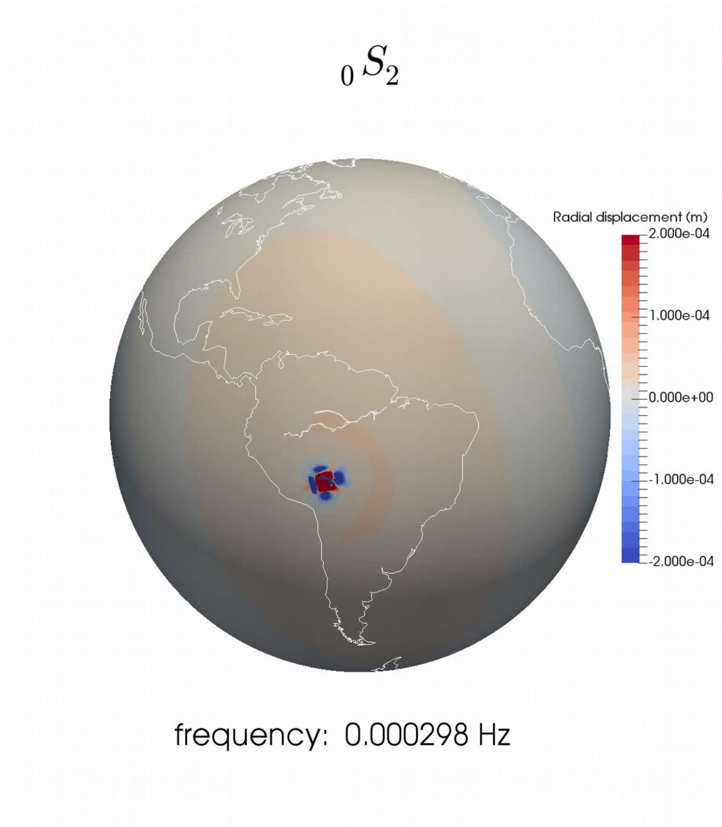
- Bolivia Earthquake
- June 09, 1994
- 8.2 Mw

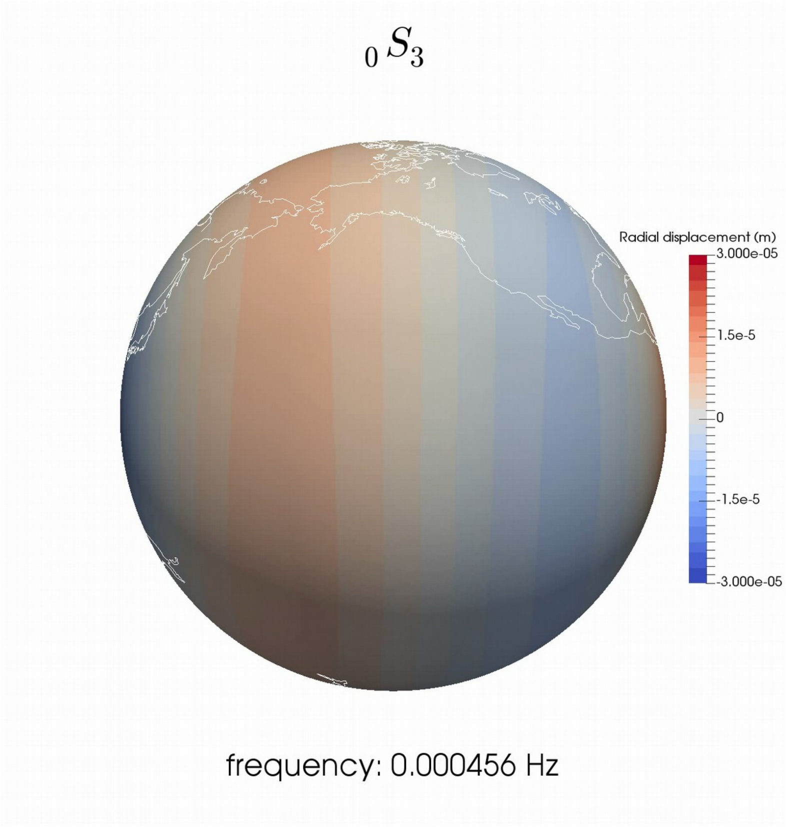


Frequency domain



Frequency domain





Conclusion

- Solution of complete governing equations
- Applications
  - Wave propagation
  - Glacial isostatic adjustment
  - Normal modes
  - Tidal loading
  - Postearthquake relaxation
- Full 3D models