Resolution Analysis By Random Probing

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"Solving an inverse problem means to describe the infinite-dimensional space of data-fitting models."

George Backus & Freeman Gilbert, 1968

1. Why resolution analysis is becoming more and more difficult

A simple example

The Resolution Matrix

$$\mathbf{R}\ \mathbf{m}_{true} = \mathbf{m}_{est}$$

Resolution matrix How the true Earth is smeared into an image. Dimension N × N.

True Earth model Dimension N.

Estimated Earth model Dimension N.

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  - ➢ We can construct bigger and bigger models $\mathbf{m}_{est}$.

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The problem:

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- As data volumes and computing power grow:
  - We can construct bigger and bigger models $\mathbf{m}_{est}$.
  - We loose our ability to quantify the quality of $\mathbf{m}_{est}$.

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Objectives of this Webinar:

- Describe 2 methods to quantify resolution when **R** is too expensive to compute and too big to store.
- One method for linear problems, and one for (mildly) nonlinear problems.
- Both based on random probing techniques.

2. Estimating the number of resolved parameters

tr **R**

Estimating The Number Of Resolved Parameters

m_i

- random test model vector
- Expectation: $E[m_i]=0$
- Covariance: $cov(m_i,m_j)=\delta_{ij}$ [uncorrelated components]

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R_{ij}

- A resolution matrix

Too large to computer.
Too large to store.
Too large to comprehend fully.

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$$E[m_i R_{ij} m_j] = R_{ij}\, E[m_i m_j]$$
$$= R_{ij} \left(E[m_i]E[m_j] + cov(m_i,m_j) \right)$$

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Hutchinson's method [Hutchinson, 1990]

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<u>Very simple recipe</u>:

- Choose a random test model **m**.
- Try to recover this model in a synthetic inversion [i.e. compute $\mathbf{m}_{est} = \mathbf{Rm}$].
- Multiply the result with **m** itself: $\mathbf{m}^T \mathbf{m}_{est} = \mathbf{m}^T\mathbf{Rm}$.
- Average over some random realisations.

- The resolution matrix itself never has to be computed!

Other Random Probing Techniques

Hutchinson, M. F. (1990), A stochastic estimator of the trace of the influence matrix for Laplacian smoothing splines, Comm. Stat. Sim., 19, 433–450.

An, M. (2012), A simple method for determining the spatial resolution of a general inverse problem, Geophys. J. Int., 191, 849–864.

Avron, H., and S. Toledo (2011), Randomized algorithms for estimating the trace of an implicit symmetric positive semi-definite matrix, J. Ass. Comp. Mach., 58, doi:10.1145/1944,345.

Drineas, P., R. Kannan, and M. W. Mahoney (2006), Fast Monte Carlo algorithms for matrices II: Computing a low-rank approximation to a matrix, SIAM J. Comput., 36, 158–183.

Frieze, A., R. Kannan, and S. Vempala (2004), Fast Monte Carlo algorithms for finding low-rank approximations, J. Assoc. Comput. Mach., 51, 1025–1041.

Halko, N., P. G. Martinsson, and J. A. Tropp (2011), Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions, SIAM Review, 53, 217–288.

MacCarthy, J. K., B. Borchers, and R. C. Aster (2011), Efficient stochastic estimation of the model resolution matrix diagonal and generalized cross-validation for large geophysical inverse problems, J. Geophys. Res., 116, doi: 1029/2011JB008,234.

Trampert, J., and A. Fichtner (2013), Resolution tests revisited: The power of random numbers, Geophys. J. Int., 192, 676–680.

3. Random probing for resolution analysis in tomography

Estimating position- and direction-dependent resolution lengths.

Point-Spread Functions

- Misfit χ in the vicinity of the optimal model **m**:

$$\chi(\mathbf{m}+\delta\mathbf{m}) = \chi(\mathbf{m}) + \frac{1}{2}\delta\mathbf{m}^T\mathbf{H}(\mathbf{m})\,\delta\mathbf{m}$$

Hessian operator
Inverse posterior covariance [assuming Gaussian errors]
Column: point-spread function

H is too expensive to compute and store.

- **But we can** infer properties of **H** from its application to random test models.

Random Probing Principle

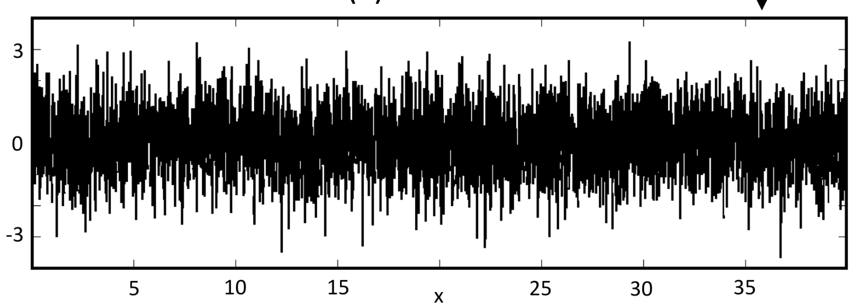
- Assume **H** is Gaussian [for simplicity and illustration]:

$$H(x;y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-y)^2}$$

$$h(y) = \int H(x;y)v(x)\,dx$$

random test model

random test model v(x)



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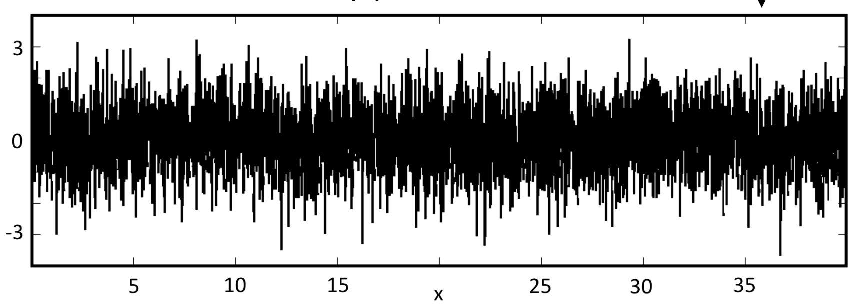
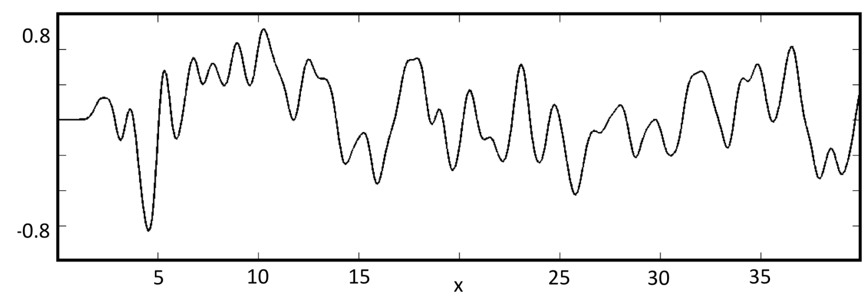


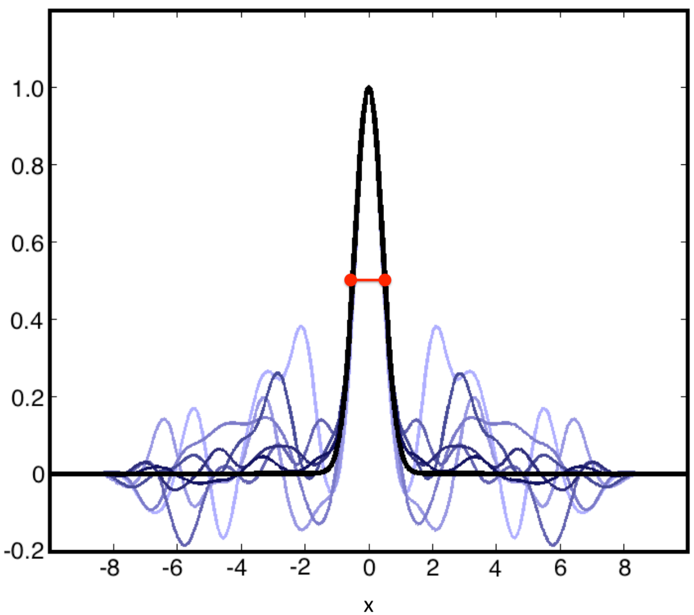
smoothed version of v(x) [h=H•v]



- Length scales of **h** contain information on length scales of **H**.

- **Auto-correlation** of the output **h** [averaged over many realisations]:

average auto-correlations of h
[for 1, 2, 3, 5, 10, 20, 50 samples]



RANDOM PROBING PRINCIPLE

- **Auto-correlation** of the output **h** [averaged over many realisations]:
- **Asymptotically**: width of auto-correlation = √2 • width of **H**

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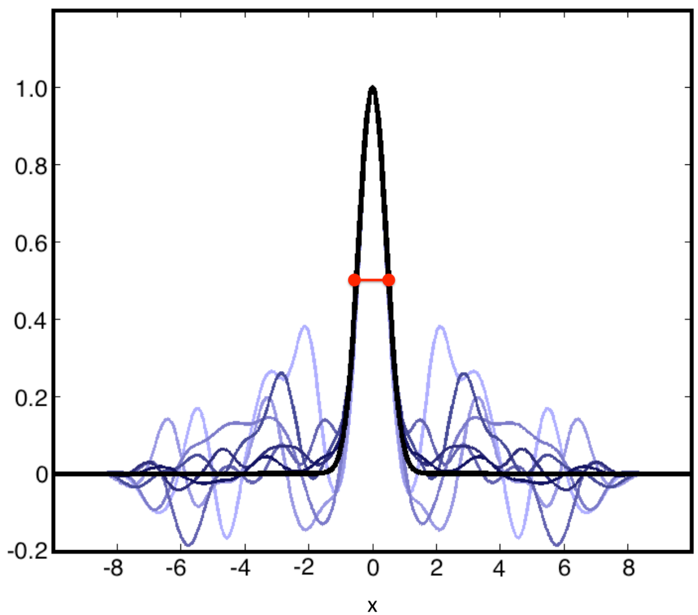

estimated with of **H**
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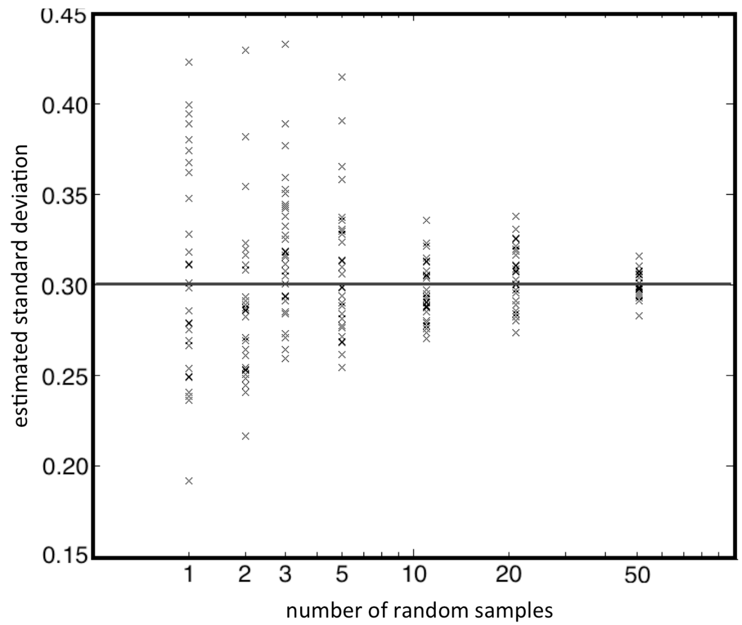


PRELIMINARY CONCLUSIONS

1. Resolution and correlations

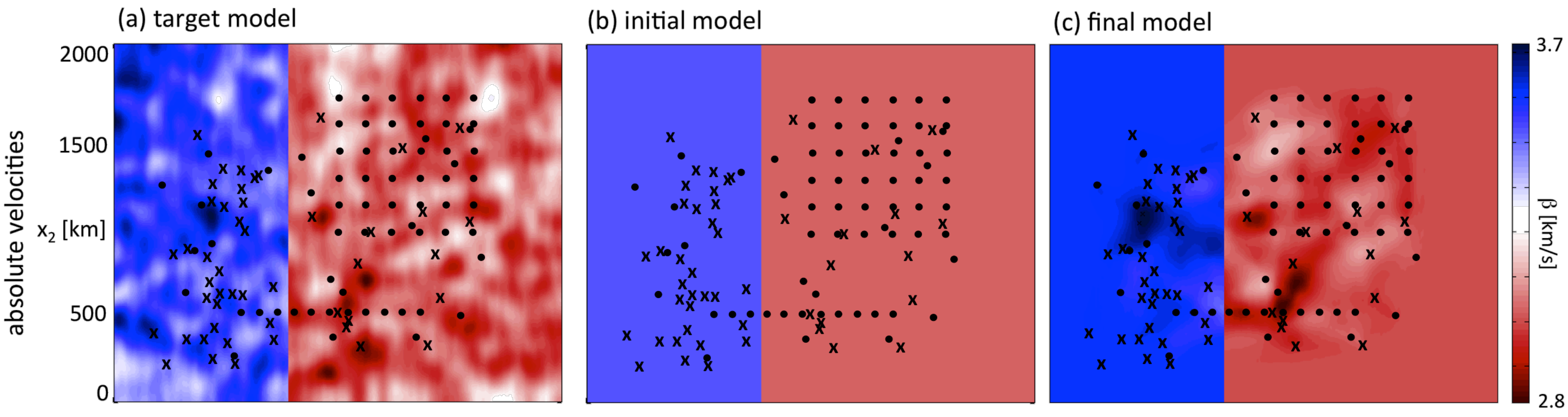
   - The Hessian acts as a smoother of random functions.
   - The smoothed functions carry information on **resolution.**
   - Can be **extracted with correlations**.

2. Convergence

   - Correlations themselves may require large sample sizes to converge.
   - The **width** of the correlation **converges extremely quickly**.
   - ➢ Useful **resolution proxies** may already be obtained with **very few samples.**

Synthetic full-waveform inversion in 2D

- Synthetic inversion setup



SYNTHETIC EXAMPLE IN 2D

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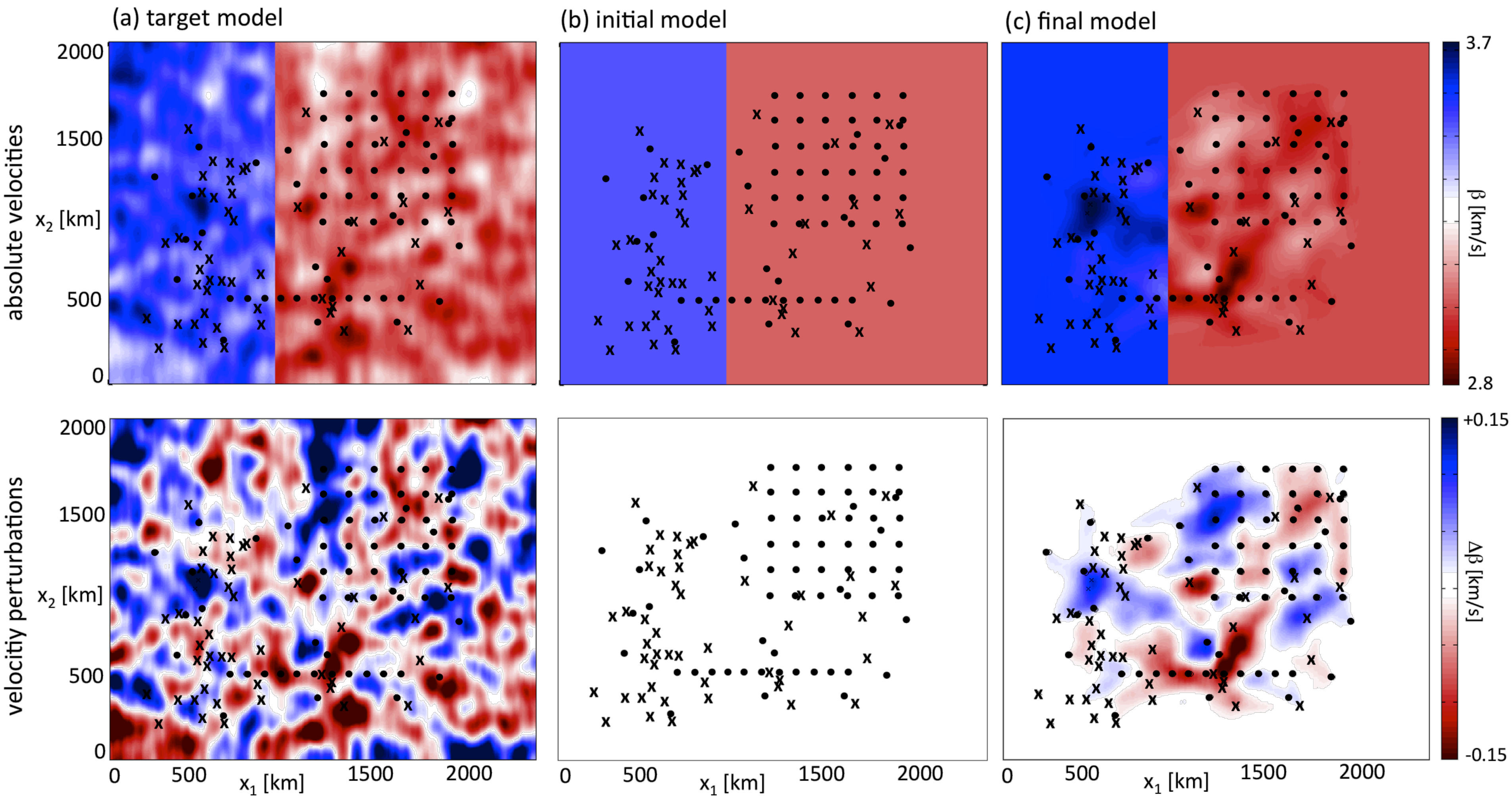

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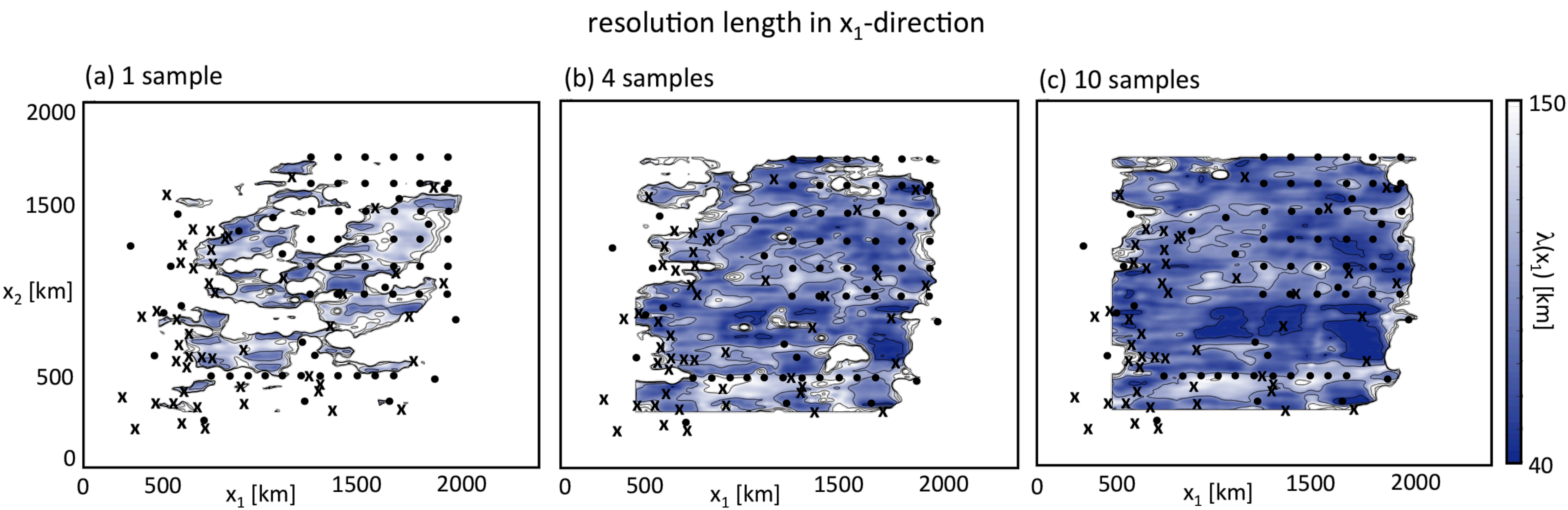
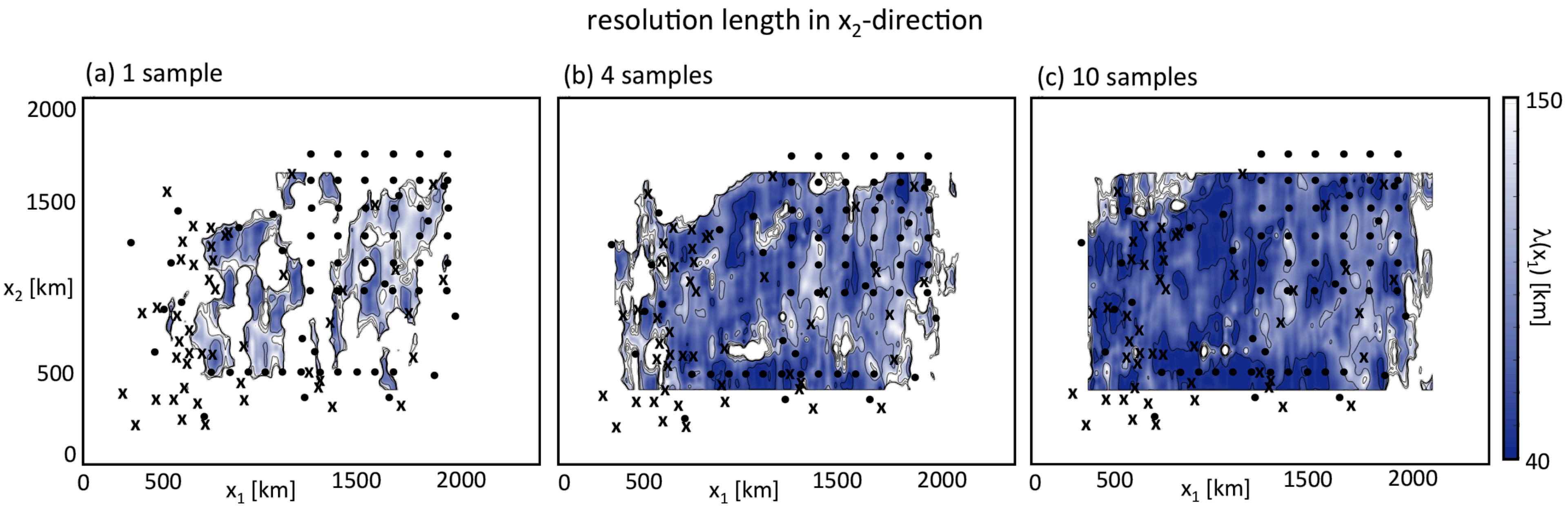
- Estimated width of the point-spread functions in **x_1-direction** [resolution length].



- Around 5-10 samples to converge.
- Resolution is strongly heterogeneous.

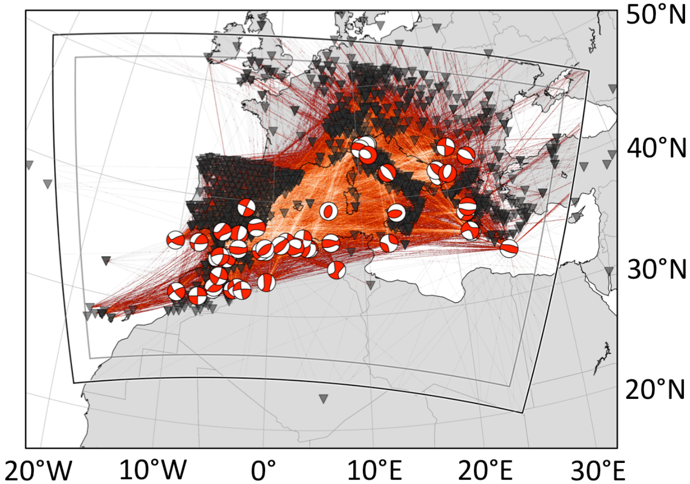
- Estimated width of the point-spread functions in **x_2-direction** [resolution length].



Real-data application

INVERSION SETUP

surface wave ray coverage



Technical summary:

Data

- 52 earthquakes, >1000 stations
- body waves, surface waves, ...
- periods: **10 – 150 s**

Forward modelling

- spectral elements
- **3D visco-elastic, anisotropic**

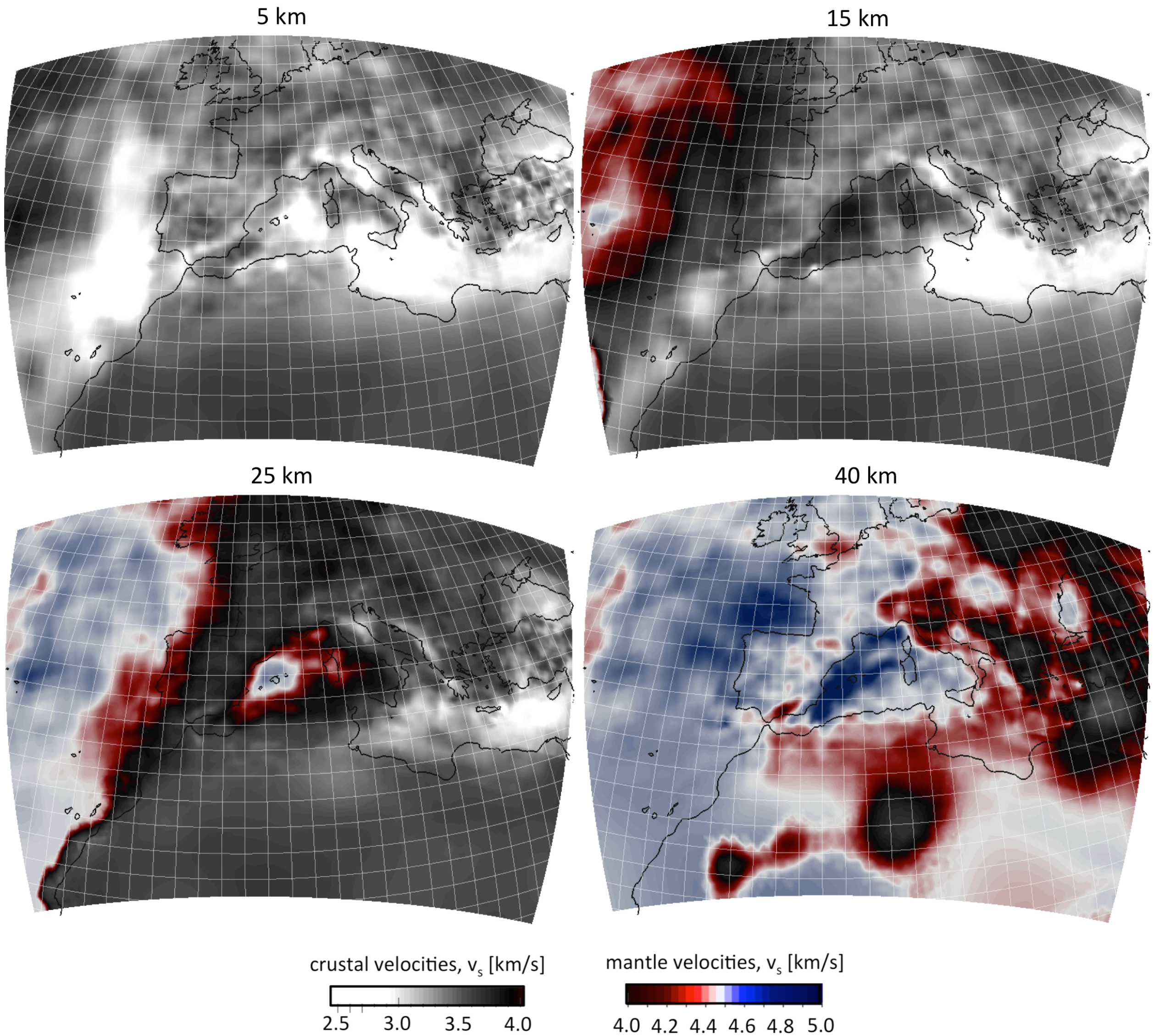
Inversion

- initial model from previous European FWI
- adjoint-based CG
- invert for v_{sh}, v_{sv}, v_p, ρ **and** source location/mechanism

Fichtner & Villasenor, EPSL 2015.

S Velocity Model

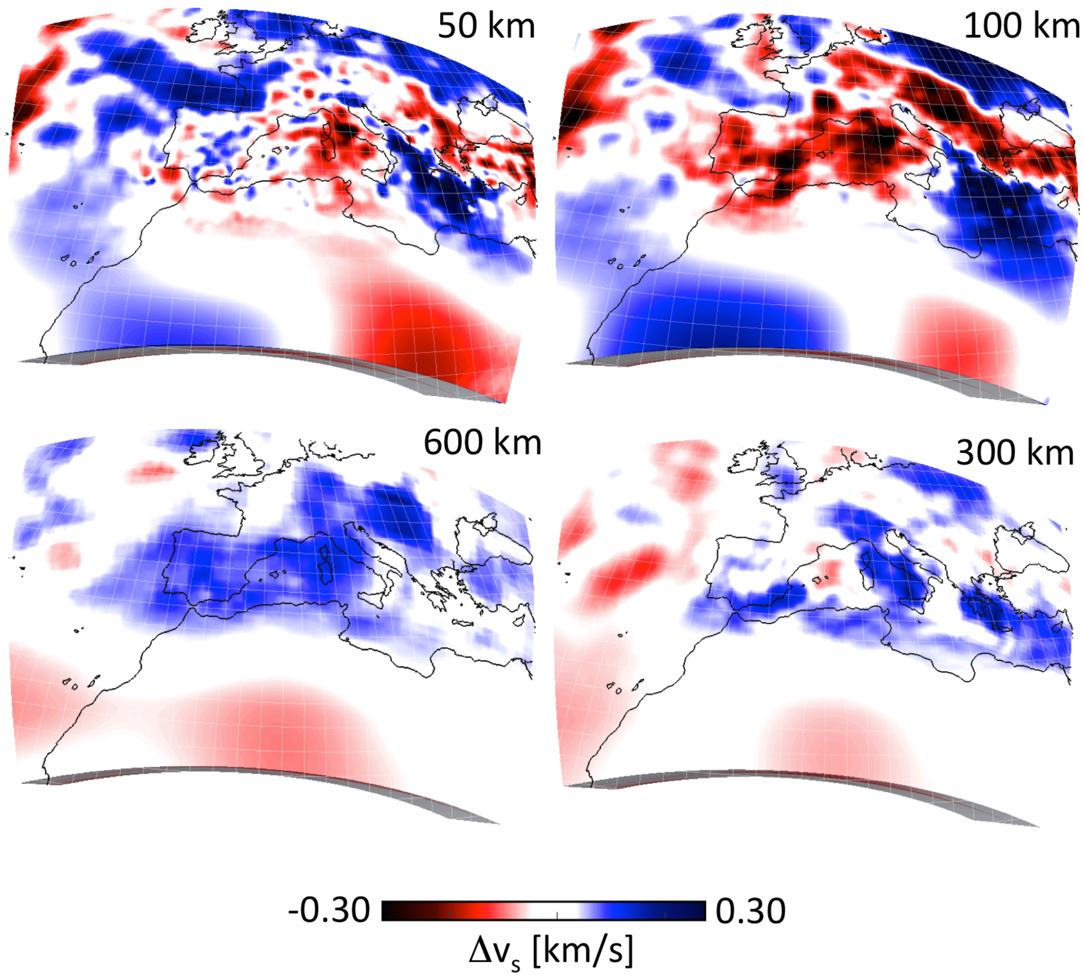
isotropic S velocity



Fichtner & Villasenor, EPSL 2015.

S Velocity Model

isotropic S velocity variations



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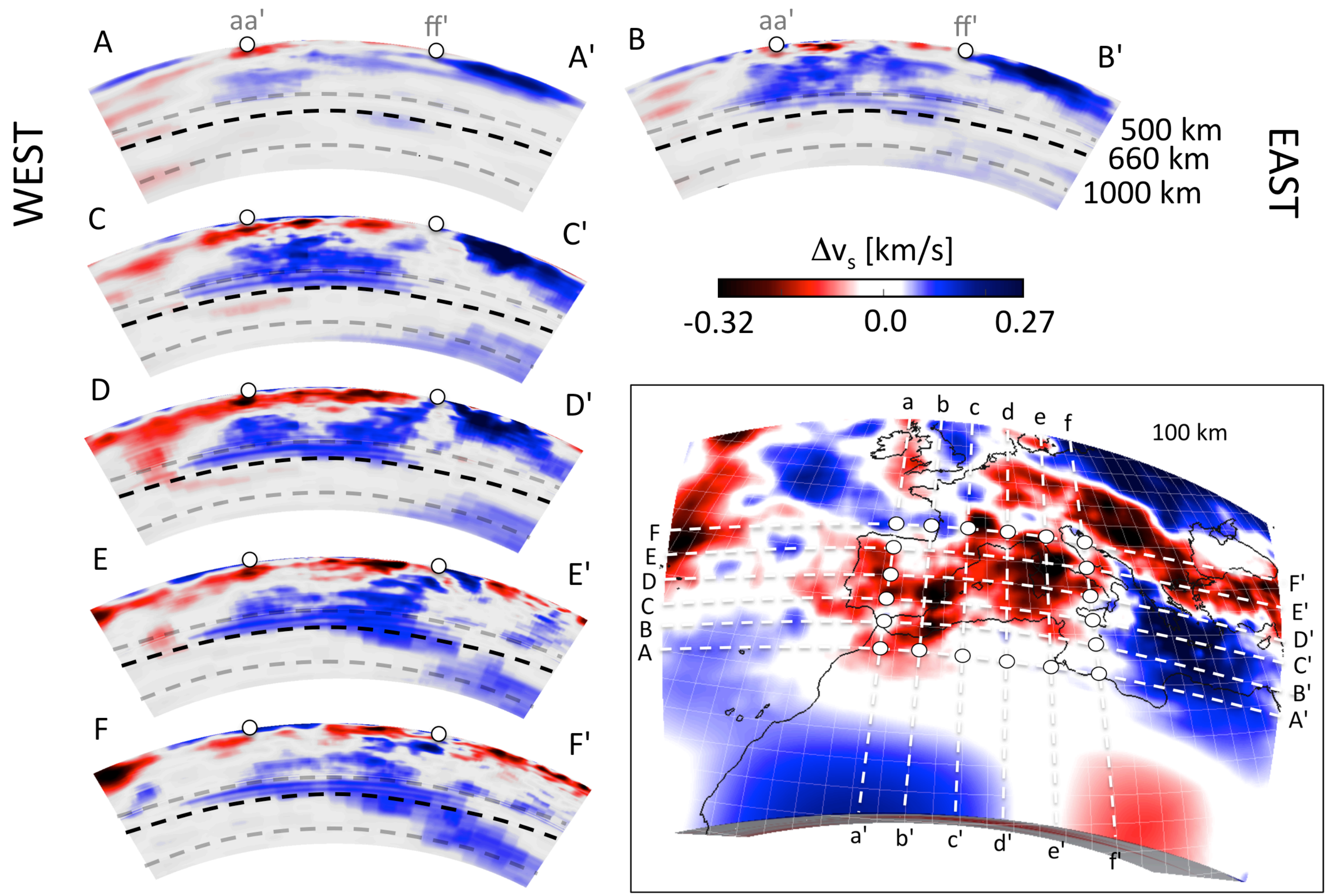

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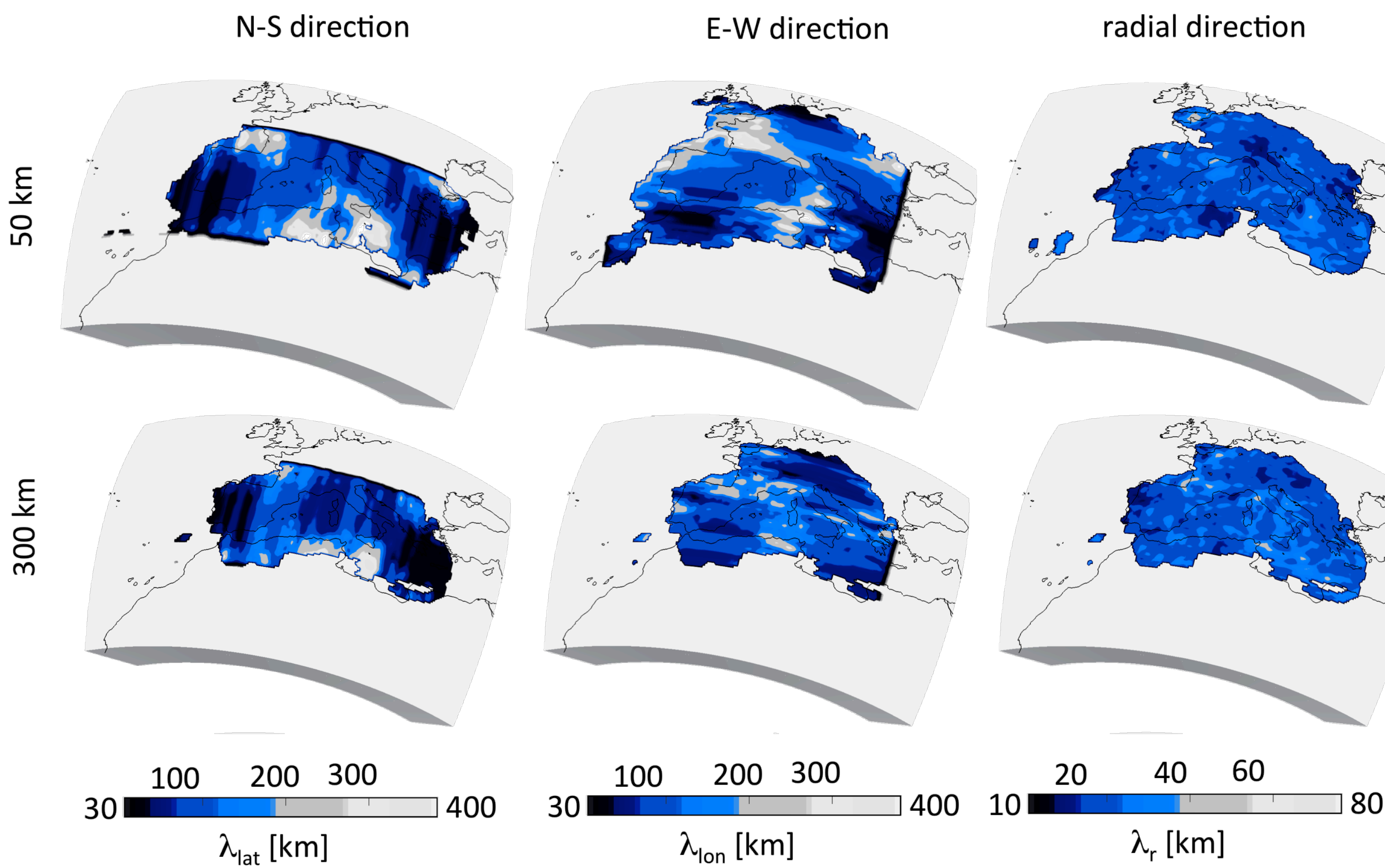
Position- and Direction-Dependent Resolution Lengths



Conclusions

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5. Applicability to any tomographic technique

Thanks for your attention!